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A differential geometric setting for BRS transformations and anomalies II

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Abstract. *Complement to our mathematical description of Becchi-Rouet-Stora relations and anomalies of gauge fields (cf. paper with the same title, part I). We treat the case where the structure group is given as a group of matrices, and provide an alternative description of the cohomology governing anomalies in terms of equivariant differential forms on the gauge group.*

INTRODUCTION

In a previous paper [1] we described in detail a differential-geometric setting for BRS transformations and anomalies considered as items of the general theory of smooth principal bundles – hence working in the context of an arbitrary smooth principal bundle $\mathbb{P} : P \to M$ whose structure group G is a general Lie group. Moreover, we constructed the cohomology relevant to anomalies as that of the Lie algebra $\mathscr{L}$ of the gauge group $\mathscr{G}$ with coefficients L – valued smooth differential forms on P, L the Lie algebra of G. We now return to the subject, in order to discuss two additional aspects absent from [1].

First, we specialize to the (usual) case of a linear structural group G (i.e. G and L are assumed to be subsets of the set M_ν of $\nu \times \nu$ complex matrices): one then naturally works with the M_ν-valued de Rham complex of P-computationally more convenient and closer to the formalism used by physicists (1).

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(1) Of e.g. [2].

As a second point we rephrase the cohomology of $\mathscr{L}$ with values in Ad-equivariant differential forms on P (carrying the pull-back representation ρ of $\mathscr{G}$) in terms of smooth ρ-equivariant differential forms on $\mathscr{G}$.

The ensuing reformulation of the structure described in [1] comes again nearer the habits of physicists [2], and displays the relationship between the cohomology operator s and the exterior derivative of $\mathscr{G}$.

This paper is written so as to be readable independently of our previous exposition [1] (to which we refer only for details of proofs). In particular we rederive from scratch the BRS relations within the present formalism.

1. THE PRINCIPAL BUNDLE $\mathbb{P} = (P \to M, G)$

Our basic object is a smooth principal bundle $\mathbb{P} = (P \to M, G)$, with total space P, base M, projection π, and structural group G. G is assumed to be a subgroup of the group of invertibles in the set M_ν of complex $\nu \times \nu$ matrices. This allows to consider G and its Lie-algebra L as embedded in M_ν, with the product of $s, s' \in G$, resp. the Lie bracket of $u, u' \in L$ obtained as ss', resp. $uu' - u'u$, where the products are performed in M_ν. We set

(1.1) $$\mathrm{ad}\, s\,(m) = sms^{-1}, \qquad m \in M_\nu, \quad s \in G,$$

and call a map $\varphi : P \to M_\nu$ *ad-equivariant* whenever one has

(1.2) $$\varphi(zs) = s^{-1}\varphi(z)s, \qquad z \in P, \qquad s \in G.$$

The *gauge group* $\mathscr{G}$, and its Lie algebra $\mathscr{L}$, then consist of the smooth ad-equivariant functions $g : P \to G$, resp. $\Omega : P \to L$, where $g \in \mathscr{G}$ acts on P as $z \in P \to gz \in P$, with

(1.3) $$gz = zg(z), \qquad z \in P$$

and with the properties $(gg')(z) = g(z)g'(z)$; $g^{-1}(z) = g(z)^{-1}$; $[\Omega, \Omega'](z) = [\Omega(z), \Omega'(z)]$; and $e^{\Omega}(z) = e^{\Omega(z)}$; $g, g' \in \mathscr{G}$, $\Omega, \Omega' \in \mathscr{L}$, $z \in P$.

2. THE M_ν-VALUED DE RHAM COMPLEX OF P

We denote by $\Lambda^p(P, \mathbb{C})$, resp. $\Lambda^p(P, M_\nu)$ the sets of smooth p-forms on P with complex values, resp. values in the complex $\nu \times \nu$ matrices: we thus have

(2.1) $$\Lambda^p(P, M_\nu) = M_\nu \otimes \Lambda^p(P, \mathbb{C})$$

with the identification:

$$(2.2)\qquad (m\otimes\alpha)(\xi_1,\ldots,\xi_p)=\alpha(\xi_1,\ldots,\xi_p)m,\quad \begin{cases} m\in M_\nu \\ \alpha\in\Lambda^p(P,\mathbf{C}), \\ \xi_i\in X(P),\ i=1,\ldots,p \end{cases}$$

($X(P)$ denotes the set of smooth vector fields on P). The *complex-valued de Rham complex of P* is the direct sum

$$(2.3)\qquad \Lambda^*(P,\mathbf{C})=\underset{p}{\oplus}\,\Lambda^p(P,\mathbf{C})$$

equipped with the *exterior differential* d. The *M_ν-valued de Rham complex* of P is

$$(2.4)\qquad \Lambda^*(P,M_\nu)=\underset{p}{\oplus}\,\Lambda^p(P,M_\nu)=M_\nu\otimes\Lambda^*(P,\mathbf{C})$$

with the differential

$$(2.5)\qquad \mathrm{d}=\mathrm{id}_{M_\nu}\otimes\mathrm{d}.$$

The groups G, resp. $\mathscr{G}$ act on $\Lambda^*(P,\mathbf{C})$ by pull back, resp. pull back of the inverse: for $\alpha\in\Lambda^P(P,\mathbf{C})$, $z\in P$, $Z_i\in T_z^P$, we set

$$(2.6)\qquad (r(s)\alpha)(z,Z_i)=\alpha(zs,Z_is),\qquad s\in G$$

$$(2.7)\qquad (\rho(g)\alpha)(z,Z_i)=\alpha(g^{-1}z,g^{-1}Z_i),\qquad g\in\mathscr{G}$$

where $Z_i\to Z_is$ and $Z_i\to g^{-1}Z_i$ denote the tangent maps at z of the maps $z\to zs$, resp. $z\to g^{-1}z$.

These actions are extended to $\Lambda^*(P,M_\nu)$ as

$$(2.8)\qquad r(s)=\mathrm{ad}s\otimes r(s),\qquad s\in G$$

$$(2.9)\qquad \rho(g)=\mathrm{id}_{M_\nu}\otimes\rho(g),\qquad g\in G$$

ρ denoting also the corresponding action of $\mathscr{L}$:

$$(2.10)\qquad \rho(\Omega)=(\mathrm{d}/\mathrm{d}t)\big|_{t=0}\rho(e^{\Omega t}),\qquad \Omega\in\mathscr{L},$$

By combining the wedge product $\wedge$ of $\Lambda^*(P,M_\nu)$ with the matrix product of M_ν, resp. its matrix commutator $[,]$, we define products $\wedge$, resp. $[\wedge]$, on $\Lambda^*(P,M_\nu)$: for $m,m'\in M_\nu$, $\alpha,\alpha'\in\Lambda^*(P,\mathbf{C})$:

$$(2.11)\qquad (m\otimes\alpha)\wedge(m'\otimes\alpha')=(mm')\otimes(\alpha\wedge\alpha')$$

$$(2.12)\qquad [m\otimes\alpha\wedge m'\otimes\alpha']=[m,m']\otimes(\alpha\wedge\alpha')$$

or alternatively, for $\mu \in \Lambda^p(P, M_\nu)$, $\mu' \in \Lambda^*(P, M_\nu)$, $\xi_i \in X(P)$:

(2.11a)
$$(\mu \wedge \mu')(\xi_1, \ldots, \xi_{p+q}) = \frac{1}{\alpha!}\,\frac{1}{\beta!}\sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma)\mu(\xi_{\sigma 1}, \ldots, \xi_{\sigma p})\mu'(\xi_{\sigma[p+1]}, \ldots, \xi_{\sigma[p+q]})$$

(2.12a)
$$[\mu \wedge \mu'](\xi_1, \ldots, \xi_{p+q}) = \frac{1}{\alpha!}\,\frac{1}{\beta!}\sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma)[\mu(\xi_{\sigma 1}, \ldots, \xi_{\sigma q}), \mu'(\xi_{\sigma[p+1]}, \ldots, \xi_{\sigma[p+q]})].$$

We denote by $A^*(P, M_\nu)$ the set of *ad-equivariant M_ν-valued differential forms on P* (= fixpoints of the action r):

(2.13) $$A^*(P, M_\nu) = \oplus A^p(P, M_\nu)$$

(2.14) $$A^p(P, M_\nu) = \{\mu \in \Lambda^p(P, M_\nu);\quad r(s)\mu = \mu \text{ for all } s \in G\}.$$

The foregoing definitions now entail the following structure:

For a p-form μ and a q-form ν, both $\mu \dot\wedge \nu$ and $[\mu \wedge \nu]$ are $(p+q)$ - forms depending linearly on μ and ν.

The product $\dot\wedge$ is associative: for $\mu, \nu, \sigma \in \Lambda^*(P, M_\nu)$:

(2.15) $$\mu \dot\wedge (\nu \dot\wedge \sigma) = (\mu \dot\wedge \nu) \dot\wedge \sigma,$$

whilst the product $[\wedge]$ has the graded Lie algebra properties: for $\mu, \nu, \sigma \in \Lambda^*(P, M_\nu)$ of degrees p, q, resp. r:

(2.16)
$$\begin{cases} [\nu \wedge \mu] = -(-1)^{pq}[\mu \wedge \nu] \\ \\ (-1)^{pr}[\mu \wedge [\nu \wedge \sigma] + (-1)^{qp}[\nu \wedge [\sigma \wedge \mu] + (-1)^{rq}[\sigma \wedge [\mu \wedge \nu]] = 0 \end{cases}$$

The linear operator d on $\Lambda^*(P, M_\nu)$ is of grade one, of square zero, and a graded derivation w.r.t. both $\dot\wedge$ and $[\wedge]$:

(2.17) $$\mathrm{d}\Lambda^p(P, M_\nu) \subset \Lambda^{p+1}(P, M_\nu)$$

(2.18) $$\mathrm{d}^2 = 0$$

(2.19) $$\mathrm{d}\{\mu \dot\wedge \nu\} = \mathrm{d}\mu \dot\wedge \nu + (-1)^p \mu \dot\wedge \mathrm{d}\nu$$

(2.20) $$\mathrm{d}[\mu \wedge \nu] = [\mathrm{d}\mu \wedge \nu] + (-1)^p[\mu \wedge \mathrm{d}\nu]$$

$$\begin{cases} \mu \in \Lambda^p(P, M_\nu) \\ \nu \in \Lambda^*(P, M_\nu) \end{cases}$$

Both products $\dot\wedge$ and $[\wedge]$ commute with all $r(s)$, $s \in G$, and $\rho(g)$, $g \in G$: for $\mu, \nu \in \Lambda^*(P, M_\nu)$

(2.21) $$r(s)\{\mu \dot{\wedge} \nu\} = r(s)\,\mu \dot{\wedge} r(s)\nu$$

(2.22) $$r(s)[\mu \wedge \nu] = [r(s)\mu \wedge r(s)\nu]$$

(2.23) $$\rho(g)\{\mu \dot{\wedge} \nu\} = \rho(g)\mu \dot{\wedge} \rho(g)\nu$$

(2.24) $$\rho(g)[\mu \wedge \nu] = [\rho(g)\mu \wedge \rho(g)\nu].$$

Accordingly $\rho(\Omega)$, $\Omega \in L$, is a zero-grade derivation for both $\dot{\wedge}$ and $[\wedge]$: for $\mu, \nu \in \Lambda^*(P, M_\nu)$

(2.25) $$\rho(\Omega)\{\mu \dot{\wedge} \nu\} = \rho(\Omega)\,\mu \dot{\wedge} \nu + \mu \dot{\wedge} \rho(\Omega)\nu$$

(2.26) $$\rho(\Omega)[\mu \wedge \nu] = [\rho(\Omega)\,\mu \wedge \nu] + [\mu \wedge \rho(\Omega)\nu].$$

One has furthermore the commutation properties

(2.27) $$r(s)\mathrm{d} = \mathrm{d}r(s), \qquad s \in G$$

(2.28) $$\rho(g)\mathrm{d} = \mathrm{d}r(g), \qquad g \in \mathscr{G}$$

(2.29) $$r(s)\rho(g) = \rho(g)r(s), \qquad s \in G,\ g \in \mathscr{G}.$$

We subsume as follows the foregoing properties:

$(\Lambda^*(P, M_\nu), \mathrm{d}, \dot{\wedge})$ is a associative GDA (2); and r, resp. ρ, are mutually commuting representations of the groups G, resp. $\mathscr{G}$, by zero-grade automorphisms of this GDA

$(\Lambda^*(P, M_\nu), \mathrm{d}, [\wedge])$ is a DGL (3), with r resp. ρ, mutually commuting representations of the groups G, resp. $\mathscr{G}$, by zero-grade automorphisms of this DGL.

As a consequence we have that

$(A^*(P, M_\nu), \mathrm{d}, \dot{\wedge})$ is a sub-GDA of $(\Lambda^*(P, M_\nu), \mathrm{d}, \dot{\wedge})$; $(A^*(P, M_\nu), \mathrm{d}, [\wedge]$ is a sub-DGL of $(\Lambda^*(P, M_\nu), \mathrm{d}, [\wedge])$, with sub-DGL $(A^*(P, L), \mathrm{d}, [\wedge]$.

Moreover, both $A^*(P, M_\nu)$ and $A^*(P, L)$ are stable under the representation ρ of $\mathscr{G}$.

For the proof of these classical results we refer to, e.g. [1] $[\Lambda^*(P, M_\nu), \mathrm{d}, \dot{\wedge}]$ is in fact the skew product of the GDAs $(M_\nu, 0, .)$ and $(\Lambda^*(P, \mathbb{C}), \mathrm{d}, \wedge)$ cf. [1], Appendix A. The proofs in section 1 of [1] relative to $(\Lambda^*(P, L), \mathrm{d}, [\wedge])$ transcribe verbatim to $(\Lambda^*(P, M_\nu), \mathrm{d}, [\wedge])$.

(2) *Graded differential algebra* cf. eg. [1] Appendix A.

(3) *Differential graded Lie algebra* cf. [1] Appendix A.

Note that:

(i) The product $[\wedge]$ is obtained as a graded commutator w.r.t. the product $\dot{\wedge}$:

(2.30) $$[\mu \wedge \nu] = \mu \dot{\wedge} \nu - (-1)^{pq} \nu \dot{\wedge} \mu, \quad \begin{cases} \mu \in \Lambda^p(P, M_\nu) \\ \\ \nu \in \Lambda^q(P, M_\nu) \end{cases}$$

as immediately follows from (2.11), (2.12). Further:

(ii) The gauge group $\mathscr{G}$, its Lie algebra $\mathscr{L}$, and the tangent space $T_g^{\mathscr{G}}$, $g \in \mathscr{G}$, are all embedded in $A^\circ(P, M_\nu)$: $\mathscr{G}$ as subgroup of its group of invertibles; L as $(\Lambda^\circ(P, L), [\wedge])$; and $T_g^{\mathscr{G}}$ as $g\mathscr{L} = gA^\circ(P, L)$.
On the other hand, the set A of connections one-forms of $\mathbb{P}$ is a convex subset of $A^1(P, M_\nu)$.

For further reference, we note that

(iii) The gauge-group $\mathscr{G}$ acts as follows on $\mathscr{L}$ and on A: we have (4)

(2.31) $$\rho(g)\Omega = g \dot{\wedge} \Omega \dot{\wedge} g^{-1}, \quad g \in \mathscr{G}, \Omega \in \mathscr{L}$$

and

(2.32) $$\begin{cases} \rho(g)a = g \dot{\wedge} a \dot{\wedge} g^{-1} - \mathrm{d}g \dot{\wedge} g^{-1} \\ \\ \rho(g^{-1})a = g^{-1} \dot{\wedge} a \dot{\wedge} g + g^{-1} \dot{\wedge} \mathrm{d}g. \end{cases}$$

3. COHOMOLOGY OF $\mathscr{L}$ WITH VALUES IN $\Lambda^*(P, M_\nu)$

We denote by $\Lambda_1^{p\alpha}$ the set of alternate α-linear forms on $\mathscr{L}$ with values in $\Lambda^p(P, M_\nu)$ (5):

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(4) It is customary to omit the product signs $\dot{\wedge}$ in the r.h.s. of these formulae (the wedge products merge into ordinary products since $g \in A^\circ(P, M_\nu)$). Note that one has $\mathrm{d}g^{-1} = - g^{-1} \dot{\wedge} \mathrm{d}g \dot{\wedge} g^{-1}$.

(5) $C^\infty(M)$ denotes the algebra of complex smooth functions on M, isomophic to the algebra of fixpoints of $\Lambda^\circ(P, \mathbb{C})$ under r. Note that $\mathscr{L}$, $\Lambda^p(P, M_\nu)$ and $A^p(P, M_\nu)$ are modules over $C^\infty(M)$ (in fact $\mathscr{L}$ is a Lie-algebra over $C^\infty(M)$).

(3.1) $$\mathbb{\Lambda}_1^{p\alpha} = \Lambda^\alpha(\mathcal{L}, \Lambda^p(P, M_\nu)) = \Lambda^p(P, M_\nu) \otimes \Phi^\alpha$$

where $\Phi^\alpha = \Lambda^\alpha(\mathcal{L}, C^\infty(M))$ and the second equality (6) corresponds to the identification

(3.2) $$(\mu \otimes \varphi)(\Omega_1, \ldots, \Omega_\alpha) = \varphi(\Omega_1, \ldots, \Omega_\alpha)\mu, \quad \begin{cases} \mu \in \Lambda^p(P, M_\nu) \\ \varphi \in \Phi^\alpha \\ \Omega_1, \ldots, \Omega_\alpha \in \mathcal{L} \end{cases}$$

and the tensor product is optionally w.r.t. $C^\infty(M)$ or to $\mathbf{C}$.

We write

(3.3) $$\begin{cases} {}^n\mathbb{\Lambda}_1 = \underset{p+\alpha=n}{\oplus} \mathbb{\Lambda}_1^{p\alpha}, \\ \mathbb{\Lambda}_1^{p*} = \underset{p}{\oplus} \mathbb{\Lambda}_1^{p\alpha}, \quad \mathbb{\Lambda}_1^{*\alpha} = \underset{p}{\oplus} \mathbb{\Lambda}_1^{p\alpha} \\ \mathbb{\Lambda}_1^{**} = \underset{p,\alpha}{\oplus} \mathbb{\Lambda}_1^{p\alpha}, \quad {}^*\mathbb{\Lambda}_1 = \underset{n}{\oplus} {}^n\mathbb{\Lambda}_1, \end{cases}$$

We now equip the space $\mathbb{\Lambda}_1^{**} = {}^*\mathbb{\Lambda}_1$ (the two symbols distinguishing between the «*double grading*» (p, α) and the «*total grading*» n) (7) with the operators d, δ, s, Δ, $r(s)$, $\rho(g)$ defined as follows (8).

(3.4) $$(\mathrm{d}\,U)(\Omega_0, \ldots, \Omega_\alpha) = \mathrm{d}\{U(\Omega_0, \ldots, \Omega_\alpha)\}$$

(3.5) $$\begin{cases} (\delta\, U)(\Omega_0, \ldots, \Omega_\alpha) = \sum_{i=0}^{\alpha} (-1)^i \rho(\Omega_i)\, U(\Omega_0, \ldots, \hat{\Omega}_i, \ldots, \Omega_\alpha) \\ + \sum_{0 \leq i < j \leq \alpha} (-1)^{i+j}\, U([\Omega_i, \Omega_j], \Omega_0, \ldots, \hat{\Omega}_i, \ldots, \hat{\Omega}_j, \ldots, \Omega_\alpha) \\ U \in \mathbb{\Lambda}_1^{*\alpha}, \Omega_0, \ldots, \Omega_\alpha \in \mathcal{L} \end{cases}$$

(3.6) $$s\, U = -(-1)^p\, \delta\, U, \quad U \in \mathbb{\Lambda}_1^{p*}$$

(3.7) $$\Delta = \mathrm{d} + S$$

(3.8) $$r(s) = r(s) \otimes \mathrm{id}_{\Phi^*}, \quad s \in G,$$

(3.9) $$\rho(g) = \rho(g) \otimes \mathrm{id}_{\Phi^*}, \quad g \in \mathcal{G}$$

(6) With appropriate completion of the tensor product . We could also require locality as in [1] (3.5).

(7) p is called the *order of form*, α the *ghost number*, n the *total grade*.

(8) $\Phi^* = \oplus\, \Phi^\alpha$. In (3.5) the caret indicates omission. There should be no confusion between the operators s and elements $s \in G$.

Moreover, we equip $^*\Lambda_1$, with the products $\dot{/\!\!\Lambda}$ resp. $[/\!\!\Lambda]$ defined as follows: for (9) $\mu \in \Lambda^*(P, M_\nu)$, $\mu' \in \Lambda^q(P, M)$, $\varphi \in \Phi^\alpha$, $\varphi' \in \Phi^*$:

(3.10) $$(\mu \otimes \varphi) \dot{/\!\!\Lambda} (\mu' \otimes \varphi') = (-1)^{\alpha q} (\mu \dot{\Lambda} \mu') \otimes (\varphi \wedge \varphi')$$

(3.11) $$[\mu \otimes \varphi /\!\!\Lambda \mu' \otimes \varphi'] = (-1)^{\alpha q} [\mu \wedge \mu'] \otimes (\varphi \wedge \varphi')$$

or alternatively, for $U \in \Lambda_1^{\alpha q}$, $V \in \Lambda_1^{q\beta}$ and $\Omega_1, \ldots, \Omega_{\alpha+\beta} \in \mathscr{L}$:

(3.10a) $$(U \dot{/\!\!\Lambda} V)(\Omega_1, \ldots, \Omega_{\alpha+\beta}) = (-1)^{\alpha q} \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma) \\ U(\Omega_{\sigma 1}, \ldots, \Omega_{\sigma\alpha}) \dot{\Lambda} V(\Omega_{\sigma[\alpha+1]}, \ldots, \Omega_{\sigma[\alpha+\beta]})$$

(3.11a) $$[U /\!\!\Lambda V](\Omega_1, \ldots, \Omega_{\alpha+\beta}) = (-1)^{\alpha q} \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma) \\ [U(\Omega_{\sigma 1}, \ldots, \Omega_{\sigma\alpha}) \wedge V(\Omega_{\sigma[\alpha+1]}, \ldots, \Omega_{\sigma[\alpha+\beta]})]$$

The foregoing definitions now imply the following structure. The operators d, resp. ρ, s, are of grade (1, 0), resp. (0, 1):

(3.12) $$\mathrm{d} /\!\!\Lambda_1^{p\alpha} \subset /\!\!\Lambda_1^{(p+1)\alpha}, \quad \delta /\!\!\Lambda_1^{p\alpha} = s /\!\!\Lambda_1^{p\alpha} \subset /\!\!\Lambda_1^{p(\alpha+1)},$$

hence d, δ, s, Δ are of total grade 1:

(3.13) $$\mathrm{d}\, {}^n\!/\!\!\Lambda_1, \quad \delta\, {}^n\!/\!\!\Lambda_1, \quad s\, {}^n\!/\!\!\Lambda_1, \quad \Delta\, {}^n\!/\!\!\Lambda_1 \subset {}^{(n+1)}\!/\!\!\Lambda_1,$$

whilst $r(s)$ and $\rho(g)$ are of grade zero:

(3.14) $$r(s) /\!\!\Lambda_1^{p\alpha}, \quad \delta(g) /\!\!\Lambda_1^{p\alpha} \subset /\!\!\Lambda_1^{p\alpha}.$$

d and δ are commuting coboundary operators:

(3.15) $$\mathrm{d}^2 = \delta^2 = \mathrm{d}\delta - \delta\mathrm{d} = 0,$$

hence

(3.16) $$s^2 = \mathrm{d}s + s\mathrm{d} = 0$$

(3.17) $$\Delta^2 = 0.$$

d and s (and hence Δ) are graded derivations (w.r.t. the total grading n) for both products $\dot{/\!\!\Lambda}$ and $[/\!\!\Lambda]$: for $U \in {}^n\!/\!\!\Lambda_1$, $V \in {}^*\!/\!\!\Lambda_1$

(3.18) $$\mathrm{d}(U \dot{/\!\!\Lambda} V) = \mathrm{d}U \dot{/\!\!\Lambda} V + (-1)^n U \dot{/\!\!\Lambda} \mathrm{d}V$$

(9) $\wedge$ r.h.s. of (3.10), (3.11) denotes the wedge product of Φ^*.

$$s(U \dot\wedge V) = sU \dot\wedge V + (-1)^n U \dot\wedge sV \tag{3.19}$$

$$\mathrm{d}[U \wedge V] = [\mathrm{d}U \wedge V] + (-1)^n [U \wedge \mathrm{d}V] \tag{3.20}$$

$$s[U \wedge V] = [sU \wedge V] + (-1)^n [U \wedge sV] \tag{3.21}$$

The $r(s)$, $s \in G$, and $\rho(g)$, $g \in \mathcal{G}$, commute with d and δ (thus with s and Δ), and with both products $\dot\wedge$ and $[\wedge]$:

$$r(s)\mathrm{d} - \mathrm{d}r(s) = 0 = r(s)\delta - \delta r(s) \qquad , s \in G \tag{3.22}$$

$$\rho(g)\mathrm{d} - \mathrm{d}\rho(g) = 0 = \rho(g)\delta - \delta\rho(g) \qquad , g \in \mathcal{G} \tag{3.23}$$

and, for $U, V \in {}^*\Lambda_1$

$$r(s)\{U \dot\wedge V\} = r(s) U \dot\wedge r(s) V \tag{3.24}$$

$$r(s)[U \wedge V] = [r(s) U \wedge r(s) V] \qquad , s \in G \tag{3.25}$$

$$\rho(g)\{U \dot\wedge V\} = \rho(g) U \dot\wedge \rho(g) V \tag{3.26}$$

$$\rho(g)[U \wedge V] = [\rho(g) U \wedge \rho(g) V]. \qquad , g \in \mathcal{G} \tag{3.27}$$

The product $\dot\wedge$ is associative:

$$(U \dot\wedge V) \dot\wedge W = U \dot\wedge (V \dot\wedge W), \; U, V \in {}^*\Lambda_1 \tag{3.28}$$

whilst the product $[\wedge]$ has the graded Lie algebra properties: we have, for $U \in {}^k\Lambda_1$, $V \in {}^m\Lambda_1$, $W \in {}^n\Lambda_1$:

$$\begin{cases} [V \wedge U] = -(-1)^{nm} [U \wedge V] \\ \\ (-1)^{ln}[U \wedge [V \wedge W]] + (-1)^{ml}[V \wedge [W \wedge U] + (-1)^{nm}[W \wedge [U \wedge V]] = 0 \end{cases} , \tag{3.29}$$

The product $[\wedge]$ is in fact obtained as a graded commutator w.r.t. $\dot\wedge$:

$$[U \wedge V] = U \dot\wedge V - (-1)^{nm} V \dot\wedge U, \qquad \begin{cases} U \in {}^n\Lambda, \\ \\ V \in {}^m\Lambda. \end{cases} \tag{3.30}$$

The last fact follows immediately from (3.10), (3.11). For the proof of the other claims, we refer to section 3 of [1], whose proofs transpose verbatim from $\Lambda^{p\alpha}$ to $\Lambda^{p\alpha}$, effecting the replacements; $L \to M_\nu$ and $\mathbb{R} \to \mathbb{C}$.

The facts (3.12) through (3.29) can be subsumed as follows: we have that

(i) $(\Lambda_1^*, \mathrm{d}, \delta)$ is a double complex with total complex $({}^*\Lambda_1, \Delta)$

(ii) $({}^*\Lambda_1, \Delta, \dot\wedge)$ is an associative GDA; and r, resp. ρ, are mutually commutative representations of the groups G, resp. $\mathscr{G}$, by zero-grade automorphism of this GDA (10).

(iii) ($\mathbb{A}_1^{**}, \Delta, [\mathbb{A}]$) is a DGL, with r, resp. ρ, mutually commuting representations of the groups G, resp. $\mathscr{G}$, by zero-grade automorphisms this DGL (10).

Setting now

$$\text{(3.31)}\qquad \begin{aligned} &\mathbf{A}_1^{p\alpha} = \Lambda^\alpha(\mathscr{L}, A^p(P, M_\nu)) = A^p(P, M_\nu) \otimes \Phi^\alpha \\ &\mathbf{A}_1^{**} = \underset{p\alpha}{\otimes} \mathbf{A}^{p\alpha}, \qquad {}^*\mathbf{A}_1 = \underset{n}{\otimes} {}^n\mathbf{A} \end{aligned}$$

we have that

$$\text{(3.32)}\qquad \mathbf{A}_1^{p\alpha} = \{U \in \Lambda_1^{p\alpha}\,; r(s)U = U \ \text{ for all } \ s \in G\},$$

hence the foregoing results imply that

(iv) ($\mathbf{A}_1^{**}$, d, δ) is a ρ-invariant sub-complex of the double complex ($\mathbb{A}_1^{**}$, d, δ);

(v) (${}^*\mathbf{A}_1, \Delta, \mathbb{A}$) is a ρ-invariant sub-GDA of (${}^*\mathbb{A}_1, \Delta, \dot{\mathbb{A}}$) (10).

(vi) (${}^*\mathbf{A}, \Delta, [\mathbb{A}]$) is a ρ-invariant sub-DGL of (${}^*\mathbb{A}_1, \Delta, [\mathbb{A}]$) (10).

4. DIFFERENTIAL FORMS ON $\mathscr{G}$ WITH VALUES IN $\Lambda^*(P, M_\nu)$

We denote by

$$\text{(4.1)}\qquad \Lambda^*(\mathscr{G}, \Lambda^*(P, M_\nu)) = \underset{p,\alpha}{\otimes} \Lambda^\alpha(\mathscr{G}, \Lambda^p(P, M_\nu))$$

the set of smooth differential forms on $\mathscr{G}$ with values in $\Lambda^*(P, M_\nu)$: α is again the «*ghost number*» and p the *order of the form*. The *total grade* n is again defined by (11)

$$\text{(4.2)}\qquad {}^n\Lambda(\mathscr{G}, \Lambda^*(P, M_\nu)) = \underset{p+\alpha=n}{\otimes} \Lambda^\alpha(\mathscr{G}, \Lambda^p(P, M_\nu)).$$

On $\Lambda^*(\mathscr{G}, \Lambda^*(P, M_\nu))$ we define as follows the operators d, δ, s and Δ: for $S \in \Lambda^\alpha(\mathscr{G}, \Lambda^p(P, M_\nu))$, $h \in \mathscr{G}$, $Z_i \in T_h^{\mathscr{G}}$ tangent vectors to $\mathscr{G}$ at h, and $\eta_i \in X(\mathscr{G})$ smooth vector fields on $\mathscr{G}$, $i = 1, \ldots, \alpha$, we set

$$\text{(4.3)}\qquad \{\mathrm{d}S\}(h; Z_1, \ldots, Z_\alpha) = \mathrm{d}\{S(h; Z_1, \ldots, Z_i, \ldots, Z_\alpha)\}$$

(10) This also holds separately with the replacements $\Delta = \mathrm{d}$ and $\Delta = s$.

(11) Coincidence of the notation d, δ, s, $\dot{\mathbb{A}}$, $[\mathbb{A}]$ with previously introduced notation is intentional. The reason for this will appear in the next section.

(4.4)
$$\{\delta S\}(\eta_1, \dots, \eta_\alpha) = \sum_{1=0}^{\alpha} (-1)^i \eta_i \{S(\eta_0, \dots, \hat{\eta}_i, \dots, \eta_\alpha)\}$$
$$+ \sum_{0 \leq i < j \leq \alpha} (-1)^{i+j} S([\eta_i, \eta_j], \eta_0, \dots, \hat{\eta}_i, \dots, \hat{\eta}_j, \dots, \eta_\alpha)$$

(4.5)
$$sS = -(-1)^p \delta s$$

and

(4.6)
$$\Delta = \mathrm{d} + s.$$

We equip $\Lambda^*(\mathcal{G}, \Lambda^*(P, M_\nu))$ with the two following products $\dot{\wedge}$ and $[\wedge]$: for elements S, T of respective degrees of form p, q, and ghost numbers α, β, we set, for $\eta_i \in X(\mathcal{G})$, $i = 0, \dots, \alpha + \beta$

(4.7)
$$(S \dot{\wedge} T)(\eta_1, \dots, \eta_{\alpha+\beta}) = (-1)^{\alpha q} \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{\alpha+\beta}} \chi(\sigma)$$
$$S(\eta_{\sigma 1}, \dots, \eta_{\sigma\alpha}) \dot{\wedge} T(\eta_{\sigma[\alpha+1]}, \dots, \eta_{\sigma[\alpha+\beta]})$$

and

(4.8)
$$[S \wedge T](\eta_1, \dots, \eta_{\alpha+\beta}) = (-1)^{\alpha q} \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{\alpha+\beta}} \chi(\sigma)$$
$$[S(\eta_{\sigma 1}, \dots, \eta_{\sigma\alpha}) \dot{\wedge} T(\eta_{\sigma[\alpha+1]}, \dots, \eta_{\sigma[\alpha+\beta]})].$$

We define an action r of G and actions ρ_1, ρ_r of $\mathcal{G}$ on $\Lambda^*(\mathcal{G}, \Lambda^*(P, M_\nu))$ as follows: for S and the η_i as above, we set (12):

(4.9)
$$\{r(s)S\}(g; Z_1, \dots, Z_\alpha) = r(s)\{S(g; Z_1, \dots, Z_\alpha)\}, \quad s \in G$$

(4.10)
$$\begin{cases} \{\rho_l(g)S\,(\eta_1, \dots, \eta_\alpha)(h) = \rho(g)\{S(g^{-1}\eta_1, \dots, g^{-1}\eta_\alpha)(g^{-1}h)\} \\ \\ \{\rho_r(g)S\}(\eta_1, \dots, \eta_\alpha)(h) = \rho(g)\{S(\eta, g, \dots, \eta_\alpha g\}(hg) \end{cases}$$

(here $g^{-1}\eta$, resp. ηg, $g \in \mathcal{G}$, $\eta \in X(\mathcal{G})$ are the smooth fields on $\mathcal{G}$ given by

(12) $r(s)$, $s \in G$, and $\rho(g)$, $g \in \mathcal{G}$, have been defined on $\Lambda^*(P, M_\nu)$ in (2.8), (2.9) and for $g, h \in \mathcal{G}$.

$$(4.11)\qquad \begin{cases} (g^{-1}\eta)_h = g^{-1}\eta_{gh} = (L_{g^{-1}})_{*gh}\eta_{gh} \\ \\ (\eta g)_h = \eta_{hg^{-1}}g = (R_{g*})_{hg^{-1}}\eta_{hg^{-1}} \end{cases}$$

where g^{-1}, resp. g on the r.h.s. denote the tangent maps to the left translation $L_{g^{-1}}$ by g resp. the right translation R_g by g in $\mathscr{G}$). ρ_l and ρ_r can also be defined as follows: defining $\rho(g)$, $g \in \mathscr{G}$, acting on $S \in \Lambda^\alpha(\mathscr{G}, \Lambda^*(P, M_\nu))$ by

$$(4.12)\qquad \{\rho(g)S\}(h; Z_1, \ldots, Z_\alpha) = \rho(g)\{S(h; Z_1, \ldots, Z_\alpha)\},$$

$\rho(g)$ evidently commutes with the pull backs L_g^*, R_g^* by the left, resp. right translations by g in G, and we have

$$(4.10a)\qquad \begin{cases} \rho_l(g) = \rho(g)\, L^*_{g^{-1}} = L^*_{g^{-1}}\rho(g) \\ \\ \rho_s(g) = \rho(g)\, R_g^* = R_g^*\, \rho(g) \end{cases} , g \in \mathscr{G}$$

A smooth, $\Lambda^*(P, M_\nu)$ – valued α – form S on $\mathscr{G}$ is now called *left* (resp. *right*) *ρ-equivariant* whenever it is a fixpoint of ρ_l (resp. ρ_r); i.e. whenever one has, for all $g, h \in G$, $Z_1, \ldots, Z_\alpha \in T_h^{\mathscr{G}}$:

$$(4.11\,l)\qquad S(gh; gZ_1, \ldots, gZ_\alpha) = \rho(g)\, S(h; Z_1, \ldots, Z_\alpha),$$

resp.

$$(4.11\,r)\qquad S(hg; Z_1 g, \ldots, Z_\alpha g) = \rho(g^{-1})\, S(h; Z_1, \ldots, Z_\alpha).$$

The foregoing definitions now imply the following structure:

(i) $(\Lambda^*(\mathscr{G}, \Lambda^*(P, M_\nu); \mathrm{d}, \delta)$ is a double complex with total complex $({}^*\Lambda(\mathscr{G}, \Lambda^*(P, M_\nu), \Delta)$

(ii) $({}^*\Lambda(\mathscr{G}, \Lambda^*(P, M_\nu), \Delta, \dot{\wedge})$ is an associative GDA; with r, resp. ρ_l, ρ_r, mutually commutative representations of the groups G, resp. $\mathscr{G}$, by zero-grade automorphisms of this GDA.

(iii) $({}^*\Lambda(\mathscr{G}, \Lambda^*(P, M_\nu), \Delta, [\wedge])$ is a DGL; with r, resp. ρ_l, ρ_r mutually commuting representations of the groups G, resp. $\mathscr{G}$, by zero-grade automorphisms of this DGL.

5. FORMULATION OF THE COHOMOLOGY IN [3] IN TERMS OF ρ-EQUIVARIANT FORMS ON $\mathscr{G}$

It follows from (4.111, r) that ρ-equivariant forms on G are determined by their value at the unit e of $\mathscr{G}$: indeed, making $h = e$ in these formulae yields:

(5.1l) $$S(g; g\Omega_1 g, \ldots, g\Omega_\alpha) = \rho(g) S(e, \Omega_1, \ldots, \Omega_\alpha),$$

resp.

(5.1r) $$S(g; g\Omega_1 g, \ldots, g\Omega_\alpha) = \rho(g^{-1}) S(e, \Omega_1, \ldots, \Omega_\alpha)$$

for all $g \in \mathscr{G}$, $\Omega_1, \ldots, \Omega_\alpha \in \mathscr{L}$. One obtains in fact in this way a bijection between our previous set $\mathbb{A}_1^{**}$ and the left, resp. right – equivariant differential forms on G. Indeed, as suggested by (5.11, r), let us assign to each $U \in \Lambda_1^{**}$, $\alpha \in \mathbb{N}$, the smooth α – form on $\mathscr{G}$ defined by

(5.2l) $$U^l(g; g\Omega_1, \ldots, g\Omega_\alpha) = \rho(g)\, U(\Omega_1, \ldots, \Omega_\alpha),$$

resp.

(5.2r) $$U^r(g; \Omega_1 g, \ldots, \Omega_\alpha g) = \rho(g^{-1})\, U(\Omega_1, \ldots, \Omega_\alpha),$$

for $g \in \mathscr{G}$, $\Omega_i \in \mathscr{L}$, $i = 1, \ldots, \alpha$; or, what comes to the same

(5.2al) $$U^l(g, Z_1, \ldots, Z_\alpha) = \rho(g)\, U(g^{-1}Z_1, \ldots, g^{-1}Z_\alpha),$$

resp.

(5.2ar) $$U^r(g, Z_1, \ldots, Z_\alpha) = \rho(g^{-1})\, U(Z_1 g^{-1}, \ldots, Z_\alpha g^{-1})$$

for $g \in \mathscr{G}$, $Z_i \in T_g^G$, $i = 1, \ldots, \alpha$: thereby we get a left ρ-equivariant form U^l, resp. a right-equivariant form U^r: indeed, we have, for $g, h \in \mathscr{G}, Z_1, \ldots, Z_\alpha \in T_h^{\mathscr{G}}$:

(5.3l) $$U^l(gh, gZ_i) = \rho(gh)\, U((gh)^{-1} g Z_i) = \rho(g)\rho(h)\, U(h^{-1} Z_i) = \rho(g)\, U^l(h, Z_i)$$

resp.

(5.3r) $$U^r(hg, Z_i g) = \rho(hg)^{-1})\; U(Z_i g (hg)^{-1}) = \rho(g^{-1})\rho(h^{-1})\, U(Z_i h^{-1}) = \rho(g^{-1})\, U^r(h, Z_i).$$

We shall denote by $l\mathbb{A}_1^{**}$, $l\mathbf{A}_1^{**}$, $r\mathbb{A}_1^{**}$, $r\mathbf{A}_1^{*}$ the image sets

(5.4) $$l\mathbb{A}_1^{p\alpha} (\text{resp. } l\mathbf{A}_1^{p\alpha}) = \{U^l;\ U \in \mathbb{A}_1^{p\alpha} (\text{resp. } \mathbf{A}_1^{p\alpha})$$
$$r\mathbb{A}_1^{p\alpha} (\text{resp. } r\mathbf{A}_1^{p\alpha}) = \{U^r;\ U \in \mathbb{A}_1^{p\alpha} (\text{resp. } \mathbf{A}_1^{p\alpha})\ .$$

These sets reproduce within $\Lambda^*(\mathscr{G}, \Lambda^*(P, M_\nu))$ the structure of $\mathbb{\Lambda}_1^{**}$ previously described in section 3. The maps $U \to U^l$ and $U \to U^r$ indeed intertwine, in the following sense, the previous structures in sections 3 and 4 we have

$$\text{(5.5)} \qquad \begin{cases} (\mathrm{d}\, U)^l = \mathrm{d}\, U^l \\ \\ (\mathrm{d}\, U)^r = \mathrm{d}\, U^r \end{cases} \quad , U \in \mathbb{\Lambda}_1^{**}$$

$$\text{(5.6)} \qquad \begin{cases} (\delta\, U)^l = \delta\, U^l \\ \\ (\delta\, U)^r = -\,\delta\, U^r \end{cases} \quad , U \in \mathbb{\Lambda}_1^{**},$$

hence

$$\text{(5.7)} \qquad \begin{cases} (s\, U)^l = s\, U^l \\ \\ (s\, U)^r = -\, s\, U^r \end{cases} \quad , U \in \mathbb{\Lambda}_1^{**},$$

and

$$\text{(5.8)} \qquad \begin{cases} (\Delta\, U)^l = \Delta\, U^l \\ \\ (\Delta\, U)^r = (\delta - s)\, U^r \end{cases} \quad , U \in \mathbb{\Lambda}_1^{**},$$

further

$$\text{(5.9)} \qquad \begin{cases} (U \dot{\wedge} V)^l = U^l \dot{\wedge} V^l \\ \\ (U \dot{\wedge} V)^r = U^r \dot{\wedge} V^r \end{cases} \quad , U, V \in \mathbb{\Lambda}_1^{**},$$

$$\text{(5.10)} \qquad \begin{cases} [U \wedge V]^l = [U^l \wedge V]^l \\ \\ [(U \wedge V)]^r = [U^r \wedge V^r] \end{cases} \quad , U, V \in \mathbb{\Lambda}_1^{**},$$

and

$$\text{(5,11)} \qquad \begin{cases} (r(s)\, U)^l = r(s)\, U^l \\ \\ (r(s)\, U)^r = r(s)\, U^r . \end{cases}$$

We check these results. Check of (5.5): we have, for $g \in \mathcal{G}$, $Z_1, \ldots, Z_\alpha \in T_g^{\mathcal{G}}$, using (3.4), (4.3) and the first equation (3.23)

$$(5.12l) \qquad \begin{aligned}(\mathrm{d}U)^l(g; Z_1, \ldots, Z_\alpha) &= \rho(g)\{(\mathrm{d}U)(g^{-1}Z_1, \ldots, g^{-1}Z_\alpha)\}\\ &= \rho(g)\,\mathrm{d}\{U(g^{-1}Z_1, \ldots, g^{-1}Z_\alpha)\}\\ &= \mathrm{d}\{\rho(g)\,U(g^{-1}Z_1, \ldots, g^{-1}Z_\alpha)\}\\ &= \mathrm{d}\{U^l(g; Z_1, \ldots, Z_\alpha)\}\\ &= (\mathrm{d}U^l)(g; Z_1, \ldots, Z_\alpha)\end{aligned}$$

$$(5.12r) \qquad \begin{aligned}(\mathrm{d}U^r)(g; Z_1, \ldots, Z_\alpha) &= \rho(g^{-1})\{(\mathrm{d}U)(Z_1 g^{-1}, \ldots, Z_\alpha g^{-1})\}\\ &= \rho(g^{-1})\,\mathrm{d}\{U(Z_1 g^{-1}, \ldots, Z_\alpha g^{-1})\}\\ &= \mathrm{d}\{\rho(g^{-1})\,U(Z_1 g^{-1}, \ldots, Z_\alpha g^{-1})\}\\ &= \mathrm{d}\{U^r(g; Z_1, \ldots, Z_\alpha)\}\\ &= (\mathrm{d}U^r)(g; Z_1, \ldots, Z_\alpha).\end{aligned}$$

Check of (5.6): for $\Omega_0, \ldots, \Omega_\alpha \in \mathcal{L}$, using (3.5), (4.4), and denoting by $g\Omega$, resp. Ωg, $\Omega \in L$, the «fundamental fields»

$$(5.13) \qquad \left\{\begin{aligned}(g\Omega)_h &= h\Omega = (L_h)_{*e}\Omega\\ & \qquad\qquad\qquad , h \in \mathcal{G},\\ (\Omega g)_h &= \Omega h = (R_h)_{*e}\Omega\end{aligned}\right.$$

we have [13]

$$(5.14l) \qquad (\delta U)^l(g\Omega_0, \ldots, g\Omega_\alpha) = \rho(g)\{\delta U(\Omega_0, \ldots, \Omega_\alpha)\}$$

$$= \rho(g)\left\{\sum_{i=0}^{\alpha} (-1)^i \rho(\Omega_i)\, U(\Omega_0, \ldots, \hat{\Omega}_i, \ldots, \Omega_\alpha)\right.$$

$$\left. + \sum_{0 \leqslant i < j \leqslant \alpha} (-1)^{i+j}\, U([\Omega_i, \Omega_j], \Omega_0, \ldots, \hat{\Omega}_i, \ldots, \hat{\Omega}_j, \ldots, \Omega_\alpha)\right\}$$

(13) g in $\rho(g)$ has to be considered as a dummy variable.

whilst

(5.15l) $$(\delta U^l)(g\Omega_0,\dots,g\Omega_\alpha) = \sum_{i=0}^{\alpha} (-1)^i (g\Omega_i)\{U^l(g\Omega_0,\dots,g\hat{\Omega}_i,\dots,g\Omega_\alpha)$$

$$+ \sum_{0\leq i<j\leq\alpha} (-1)^{i+j} U^l([g\Omega_i, g\Omega_j], g\Omega_0,\dots,g\hat{\Omega}_i,\dots,g\hat{\Omega}_j,\dots,g\Omega_\alpha)\}$$

$$= \sum_{i=0}^{\alpha} (-1)^i (g\Omega_i)\ \rho(g)\ U(\Omega_0,\dots,\hat{\Omega}_i,\dots,\Omega_\alpha)$$

$$+ \sum_{0\leq i<j\leq\alpha} (-1)^{i+j} U(g^{-1}[g\Omega_i, g\Omega_j], \Omega_0,\dots,\hat{\Omega}_i,\dots,\hat{\Omega}_j,\dots,\Omega_\alpha)\ .$$

The first equation (5.6) then follows from the relations

(5.16l) $$[g\Omega, g\Omega'] = g[\Omega, \Omega'], \qquad \Omega, \Omega' \in \mathscr{L},\ g \in \mathscr{G}$$

and (13)

(5.17l) $$\rho(g)\rho(\Omega) = (g\Omega)\{\rho(g)\}, \qquad \Omega' \in \mathscr{L},\ g \in \mathscr{G}$$

the first of which is the definition of the Lie bracket of $\mathscr{L}$, whilst the second follows from

(5.17l) $$\rho(g)\rho(\Omega) = \mathrm{d}/\mathrm{d}t\,|_{t=0}\,\rho(g)\rho(e^{t\Omega}) = \mathrm{d}/\mathrm{d}t\,|_{t=0}\,\rho(ge^{t\Omega})$$

$$= (g\Omega)\{\rho(g)\}.$$

Analogously (13)

(5.14r) $$(\delta U)^r(\Omega_0 g, \dots, \Omega_\alpha g) = \rho(g^{-1})\{\delta U(\Omega_0, \dots, \Omega_\alpha)\}$$

$$= \rho(g^{-1})\left\{\sum_{i=0}^{\alpha} (-1)^i \rho(\Omega_i)\, U(\Omega_0, \dots, \hat{\Omega}_i, \dots, \Omega_\alpha)\right.$$

$$\left. + \sum_{0\leq i\leq j\leq\alpha} (-1)^{i+j} U([\Omega_i, \Omega_j], \Omega_0, \dots, \hat{\Omega}_i, \dots, \hat{\Omega}_j, \dots, \Omega_\alpha)\right\}$$

whilst

(5.15r)
$$(\delta U^r)(\Omega_0 g,\dots,\Omega_\alpha g) = \sum_{i=0}^{\alpha} (-1)^i (\Omega_i g)\{U^r(\Omega_0 g,\dots,\widehat{\Omega_i g},\dots,\Omega_\alpha g)\}$$

$$+ \sum_{0\leq i<j\leq\alpha} (-1)^{i+j} U^r([\Omega_i g,\Omega_j g],\Omega_0 g,\dots,\widehat{\Omega_i g},\dots,\widehat{\Omega_j g},\dots,\Omega_\alpha g)$$

$$= \sum_{i=0}^{\alpha} (-1)^i (\Omega_i g)\{\rho(g^{-1})\}\, U(\Omega_0,\dots,\hat{\Omega}_i,\dots,\Omega_\alpha)$$

$$+ \sum_{0\leq i<j\leq\alpha} (-1)^{i+j} U([\Omega_i g,\Omega_j g]g^{-1},\Omega_0,\dots,\hat{\Omega}_i,\dots,\hat{\Omega}_j,\dots,\Omega_\alpha)$$

the second equation (5.6) resulting now from

(5.16r)
$$[\Omega g,\Omega' g] = -[\Omega,\Omega']g, \qquad \Omega,\Omega'\in\mathscr{L},\ g\in\mathscr{G}$$

and (13)

(5.17r)
$$\rho(g^{-1})\rho(\Omega) = -(\Omega g)\ \rho(g^{-1}), \qquad \Omega\in\mathscr{L},\ g\in\mathscr{G}$$

indeed, one has

(5.18r)
$$\rho(g^{-1})\rho(\Omega) = \mathrm{d}/\mathrm{d}t\,|_{t=0}\,\rho(g^{-1})\rho(e^{t\Omega}) = \mathrm{d}/\mathrm{d}t\,|_{t=0}\,\rho(g^{-1}e^{t\Omega})$$
$$= -\,\mathrm{d}/\mathrm{d}t\,|_{t=0}\,\rho((e^{t\Omega}g)^{-1}) = -(\Omega g)\ \rho(g^{-1})$$

Relations (5.7) and (5.8) immediately follow from (5.6) and from the patent fact that degree of form, ghost number, and total grading are the same for $U\in\mathbb{A}_1^{**}$, U^l and U^r.

Check of (5.9): we have, for $U\in\mathbb{A}_1^{*\alpha}$, $V\in\mathbb{A}_1^{q*}$ and $\Omega_1,\dots,\Omega_{\alpha+\beta}\in\mathscr{L}$, by (4.7) and (3.10a), using (2.23)

(5.19)
$$(U^l\,\dot{\wedge}\,V^l)(g;g\Omega_1,\dots,g\Omega_{\alpha+\beta}) = (-1)^{\alpha q}\frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{p+q}}\chi(\sigma)$$
$$U^l(g;g\Omega_{\sigma 1},\dots,g\Omega_{\sigma\alpha})\,\dot{\wedge}\,V^l(g:g\Omega_{\sigma[\alpha+1]},\dots,g\Omega_{\sigma[\alpha+\beta]})$$
$$= (-1)^{\alpha q}\frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{p+q}}\chi(\sigma)$$
$$\rho(g)\{U(\Omega_{\sigma 1},\dots,\Omega_{\sigma\alpha})\,\dot{\wedge}\,V(\Omega_{\sigma[\alpha+1]},\dots,\Omega_{\sigma[\alpha+\beta]})\}$$
$$= \rho(g)(U\,\dot{\wedge}\,V)(\Omega_1,\dots,\Omega_{\alpha+\beta})$$
$$= (U\,\dot{\wedge}\,V)^l\,(g;g\Omega_1,\dots,g\Omega_{\alpha+\beta})$$

and analogously

$$(5.19r)\qquad (U^r \dot{\wedge} V^r)(g, \Omega_1 g,\ldots, g\Omega_{\alpha+\beta} g) = (-1)^{\alpha q}\frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{p+q}}\chi(\sigma)$$

$$U^r(g; g\Omega_{\sigma 1} g,\ldots, \Omega_{\sigma\alpha} g)\dot{\wedge} V^r(g;\Omega_{\sigma[\alpha+1]} g,\ldots, g\Omega_{\sigma[\alpha+\beta]} g)$$

$$= (-1)^{\alpha q}\frac{1}{\alpha!}\frac{1}{\beta!}\sum_{\sigma\in\Sigma_{p+q}}\chi(\sigma)$$

$$\rho(g^{-1})\{U(\Omega_{\sigma 1},\ldots,\Omega_{\sigma\alpha})\dot{\wedge} V_{[\Omega\,\sigma[\alpha+1]]},\ldots,\Omega_{\sigma[\alpha+\beta]})\}$$

$$= \rho(g^{-1})(U\dot{\wedge} V)(\Omega_1,\ldots,\Omega_{\alpha+\beta})$$

$$= (U\dot{\wedge} V)^r(g;\Omega_1 g,\ldots,\Omega_{\alpha+\beta} g).$$

Check of (5.10): proof as in the preceding proof of (5.9), with the replacements: $\dot{\wedge} \to [\dot{\wedge}]$, using the definitions (4.8) and (3.11a), and property (2.24)

The map $U \to U^l$ (resp. $U \to U^r$) is an injective homorphism for the six following pairs of structures:

– (i) from the double complex $(\Lambda_1^{**}, \mathrm{d}, \delta)$ to the double complex $(\Lambda^*(\mathcal{G}, \Lambda^*(P, M_\nu), \mathrm{d}, \delta)$ (resp. $\Lambda^*(\mathcal{G}, \Lambda^*(P, M_\nu), \mathrm{d}, -\delta))$; with range the sub-double complex $(l\Lambda_1^*, \mathrm{d}, \delta)$ (resp. $(r\Lambda_1^*, \mathrm{d}, -\delta))$ (14).
– (ia) from the double complex $(\mathbf{A}_1^{**}, \mathrm{d}, \delta)$ to the double complex $(\Lambda^*(\mathcal{G}, A^*(P, M_\nu), \mathrm{d}, \delta)$ (resp. $\Lambda^*(\mathcal{G}, A^*(P, M_\nu), \mathrm{d}, -\delta))$; with range the sub-double complex $(l\mathbf{A}_1^*, \mathrm{d}, \delta)$ (resp. $(r\mathbf{A}_1^*, \mathrm{d}, -\delta))$.
– (ii) from the associative GDA $(^*\Lambda_1, \Delta, \dot{\wedge})$ to the associative GDA $(^*\Lambda(\mathcal{G}, \Lambda^*(P, M_\nu), \Delta, \dot{\wedge})$ (resp. $(^*\Lambda(\mathcal{G}, \Lambda^*(P, M_\nu), \mathrm{d}-s, \dot{\wedge}))$; with range the sub-GDA $(l^*\Lambda_1, \Delta, \dot{\wedge})$ (resp. $(r^*\Lambda_1, \mathrm{d}-s, \dot{\wedge}))$ (14), (15).
– (iia) from the associative GDA $(^*\mathbf{A}_1, \Delta, \dot{\wedge})$ to the associative GDA $(\Lambda^*(\mathcal{G}, A^*(P, M_\nu), \Delta, \dot{\wedge})$ (resp. $(^*\Lambda(\mathcal{G}, A^*(P, M_\nu), \mathrm{d}-s, \dot{\wedge}))$; with range the sub-GDA $(l^*\mathbf{A}_1, \Delta, \dot{\wedge})$ (resp. $(r^*\mathbf{A}_1, \mathrm{d}-s, \dot{\wedge}))$ (14), (15).
– (iii) from the DGL $(^*\Lambda_1, \Delta, [\dot{\wedge}])$ to the DGL $(^*\Lambda(\mathcal{G}, \Lambda^*(P, M_\nu), \Delta, [\dot{\wedge}])$ (resp. $\Lambda^*(\mathcal{G}, \Lambda^*(P, M_\nu), \mathrm{d}-s, [\dot{\wedge}]))$; with range the sub-DGL $(l^*\Lambda_1, \Delta, [\dot{\wedge}])$ (resp. $(r^*\Lambda_1, \mathrm{d}-s, [\dot{\wedge}]))$ (14), (15).

(14) Intertwining the representations r of G on its domain and range.

(15) Holds separately for the contributions d, s of Δ on the domain and range of $U \to U^l$ (resp. for the contributions d, s of Δ on the domain, and d, $-s$ of $\mathrm{d}-s$ on the range of $U \to U^r$).

– (iiia) from the DGL $({}^*\mathbf{A}_1, \Delta, [\wedge])$ to the DGL $({}^*\Lambda(\mathscr{G}, A^*(P, M_\nu), \Delta, [\wedge])$ (resp. $\Lambda^*(\mathscr{G}, A^*(P, M_\nu), \mathrm{d} - s, [\wedge]))$; with range the sub-DGL $(l^*\mathbf{A}_1, \Delta, [\wedge])$ (resp. $(r^*\mathbf{A}_1, \mathrm{d} - s, [\wedge]))$ (14), (15).

6. EQUIVARIANT VERSION OF THE BRS RELATIONS

In part [1] we established the BRS-relations

$$(6.1)\qquad \begin{cases} sa = -\,\mathrm{d}\,\omega - [a \wedge \omega] \\ \\ s\omega = -\,(1/2)\,[\omega \wedge \omega] \end{cases}$$

where a is the one-form of any principal connection on $\mathbf{P}$, and ω the Maurer-Cartan form of $\mathscr{G}$, both considered as elements of the DGL $({}^*A_1, \Delta = \mathrm{d} + s, [\wedge])$ (16): this was obtained as follows: one has naturally (17) $a \in \mathbf{A}_1^{10}$ and interprets ω as an element of $\mathbf{A}_1^{01}$ as the identity map of $\mathscr{L}$, thus a linear map from $\mathscr{L}$ to $A^0(P, L) = \mathscr{L}$.

As element of ${}^*\mathbf{A}_1$, a and ω give rise to left (resp. right) ρ-equivariant forms a^l, ω^l (resp. a^r, ω^r) on $\mathscr{G}$. Using the homomorphism [5] (iiia) – specifically the relations (5.5), (5.7), (5.10) – (6.1) is turned into

$$(6.1l)\qquad \begin{cases} sa^l = -\,\mathrm{d}\,\omega^l - [a^l \wedge \omega^l] \\ \\ s\omega^l = -\,(1/2)\,[\omega^l \wedge \omega^l] \end{cases}$$

resp.

$$(6.1\mathrm{r})\qquad \begin{cases} sa^r = +\,\mathrm{d}\,\omega^r + [a^r \wedge \omega^r] \\ \\ s\omega^r = +\,1/2\,[\omega^r \wedge \omega^r] \end{cases}$$

We shall now rederive these equivariant versions of the BRS relations within $\Lambda^*(\mathscr{G}, \Lambda^*(P, M_\nu))$, using the analytical apparatus of section 4 (since the corresponding computation is not merely a transcription of our former derivation in [1], revealing instead in an instructive way the role of δ as the exterior derivative of $\mathscr{G}$)

(16) We recall that $({}^*\mathbb{A}_1, [\wedge])$ and $({}^*\mathbf{A}_1, [\wedge])$ are DGLs with either Δ, d, or s as derivation.

(17) $\Lambda^1(P, M_\nu)$ being naturally embedded in $\mathbf{A}_1^{10}$ as $\Lambda^1(P, M_\nu) \otimes 1_{\Phi^*}$.

Our first task is to work out the equivariant versions a^l, ω^l (resp. a^r, ω^r) of a and ω: this is done by using the definitions of section 4 as explicit in the following remarks.

Remark 1. Denote by g the identity map of $\mathscr{G}$:

$$\mathrm{g}(h) = h, \qquad g \in \mathscr{G}, \tag{6.2}$$

considered as belonging to $\Lambda^0(\mathscr{G}, \Lambda^0(P, M_\nu))$ as id : $\mathscr{G} \to \mathscr{G} = A^0(P, M_\nu)$ (18). We have

$$(\mathrm{d}\mathrm{g})(h) = \mathrm{d}h, \qquad h \in \mathscr{G} \tag{6.3}$$

and

$$(\delta \mathrm{g})(h, Z) = Z, \qquad h \in \mathscr{G},\ Z \in T_h^{\mathscr{G}} \tag{6.4}$$

with $\delta \mathrm{g} \in \Lambda'(G, \Lambda^0(P, M_\nu)$ in as much as $T_h^{\mathscr{G}} = g\mathscr{L} \subset gA^0(P, M_\nu)$.

Indeed, definitions (3.3) and (3.5) imply respectively

$$(\mathrm{d}\mathrm{g})(h) = \mathrm{d}\ \mathrm{g}(h) = \mathrm{d}h, \qquad h \in \mathscr{G}, \tag{6.5}$$

and

$$\begin{aligned}(\delta \mathrm{g})(g\Omega)(h) &= (g\Omega)(\mathrm{g})\ (h) = \mathrm{d}/\mathrm{d}t\,|_{t=0}\mathrm{g}(he^{t\Omega}) \\ &= \mathrm{d}/\mathrm{d}t\,|_{t=0} he^{t\Omega} = h\Omega.\end{aligned} \tag{6.6}$$

Remark 2. We have (19)

$$a^l = \mathrm{g} \dot{\wedge} a \dot{\wedge} \mathrm{g}^{-1} - \mathrm{d}\mathrm{g} \dot{\wedge} \mathrm{g}^{-1} \tag{6.7l}$$

$$a^r = \mathrm{g}^{-1} \dot{\wedge} a \dot{\wedge} \mathrm{g} + \mathrm{g}^{-1} \dot{\wedge} \mathrm{d}\mathrm{g} \tag{6.7r}$$

where $a \in \mathbf{A}_1^{10}$ in the r.h.s. (20). On the other hand

$$\omega^l = \delta \mathrm{g} \dot{\wedge} \mathrm{g}^{-1} \tag{6.8l}$$

$$\omega^r = \mathrm{g}^{-1} \dot{\wedge} \delta \mathrm{g}. \tag{6.8r}$$

Using the definition of the product $\dot{\wedge}$ (cf. (3.10a)), and (6.2), (6.3), the values

(18) Observe that $\mathrm{g} \in \Lambda_1^{00}$ does not belong to $\mathbf{A}_1^{00}$.

(19) The signs $\dot{\wedge}$ in (6.7l, r) could be omitted on the ground that, since $\mathrm{g}, \mathrm{g}^{-1} \in \Lambda^0(\mathscr{G}, \Lambda^0(P, M_\nu)$, wedge products on G and P merge into usual matrix products in M_ν.

(20) Cf. footnote (18).

$a^l(g)$ and $a^r(g)$ of a^l, a^r as defined in (6.7l, r) are readily found to coincide with the r.h.s. in formulae (2.32), as should be the case since

$$\text{(6.9)} \qquad \begin{cases} a^l(g) = \rho(g)a \\ \\ a^r(g) = \rho(g^{-1})a \end{cases}, g \in \mathscr{G}$$

according to definitions (5.2l, r).

On the other hand we have, from (4.11l, r), for $Z \in T_g^{\mathscr{G}}$, $g \in \mathscr{G}$

$$\text{(6.10)} \qquad \begin{cases} \omega^l(g, Z) = \rho(g)\omega(g^{-1}Z) = \rho(g)(g^{-1}Z) = Zg^{-1} \\ \\ \omega^r(g, Z) = \rho(g^{-1})\omega(Zg^{-1}) = \rho(g^{-1})(Zg^{-1}) = g^{-1}Z \end{cases}$$

whence (6.8l, r), using (6.4) and (4.7).

Remark 3. Writing

$$\text{(6.11)} \qquad \begin{cases} a^l = a_0^l + \check{\omega}^l \\ \\ a^r = a_0^r + \check{\omega}^r \end{cases}$$

where

$$\text{(6.12)} \qquad \begin{cases} a_0^l = \mathrm{g} \dot{\wedge} a \dot{\wedge} \mathrm{g}^{-1} \\ \\ a_0^r = \mathrm{g}^{-1} \dot{\wedge} a \dot{\wedge} \mathrm{g} \end{cases}$$

and

$$\text{(6.13)} \qquad \begin{cases} \check{\omega}^l = -\,\mathrm{dg} \dot{\wedge} \mathrm{g}^{-1} \\ \\ \check{\omega}^r = +\,\mathrm{g}^{-1} \dot{\wedge} \mathrm{dg} \end{cases}$$

we have

$$\text{(6.14)} \qquad \begin{cases} sa_0^l = -\,[a_0^l \dot{\wedge} \omega^l] \\ \\ sa_0^r = [a_0^r \dot{\wedge} \omega^r] \end{cases}$$

and

$$(6.15)\qquad \begin{cases} s\breve{\omega}^l = -\,\mathrm{d}\omega^l - [\breve{\omega}^l \wedge \omega^l] \\ \\ s\breve{\omega}^r = \mathrm{d}\omega^r + [\breve{\omega}^r \wedge \omega^r]. \end{cases}$$

Proof. We have, using (3.19) with 4(ii), the ensuing fact that

$$(6.16)\qquad sg^{-1} = -\,g^{-1} \dot\wedge sg \dot\wedge g = g^{-1} \dot\wedge \delta g \dot\wedge g^{-1}$$

(cf. (4.5)), the fact that $sa = 0$, and 4(iii),

$$\begin{aligned}(6.17l)\qquad sa_0^l &= sg \dot\wedge a \dot\wedge g^{-1} - g \dot\wedge a \dot\wedge sg^{-1} \\ &= -\,\delta g \dot\wedge g^{-1} \dot\wedge g \dot\wedge a \dot\wedge g^{-1} - g \dot\wedge a \dot\wedge g^{-1} \dot\wedge \delta g \dot\wedge g^{-1} \\ &= -\,\omega^l \dot\wedge a_0^l - a_0^l \dot\wedge \omega^l = -\,[a_0^l \wedge \omega^l]\end{aligned}$$

$$\begin{aligned}(6.17r)\qquad sa_0^r &= sg^{-1} \dot\wedge a \dot\wedge g - g^{-1} \dot\wedge a \dot\wedge sg \\ &= g^{-1} \dot\wedge \delta g \dot\wedge g^{-1} \dot\wedge a \dot\wedge g + g^{-1} \dot\wedge a \dot\wedge g \dot\wedge g^{-1} \dot\wedge \delta g \\ &= \omega^r \dot\wedge a_0^r + a_0^r \dot\wedge \omega^r = +\,[a_0^r \wedge \omega^r].\end{aligned}$$

On the other hand, one has, using also (3.18) with 4(ii):

$$\begin{aligned}(6.18l)\qquad s\breve{\omega}^l &= -\,s\,\mathrm{d}g \dot\wedge g^{-1} + \mathrm{d}g \dot\wedge sg^{-1} \\ &= -\,\mathrm{d}\delta g \dot\wedge g^{-1} + \mathrm{d}g \dot\wedge g^{-1} \dot\wedge \delta g \dot\wedge g^{-1} \\ &= -\,\mathrm{d}\omega^l - \delta g \dot\wedge g^{-1} \dot\wedge \mathrm{d}g \dot\wedge g^{-1} - \breve{\omega}^l \dot\wedge \omega^l \\ &= -\,\mathrm{d}\omega^l - \omega^l \dot\wedge \breve{\omega}^l - \breve{\omega}^l \dot\wedge \omega^l \\ &= -\,\mathrm{d}\omega^l - [\omega^l \dot\wedge \breve{\omega}^l]\end{aligned}$$

$$\begin{aligned}(6.18r)\qquad s\breve{\omega}^r &= -\,sg^{-1} \dot\wedge \mathrm{d}g + g^{-1} \dot\wedge s\,\mathrm{d}g \\ &= -\,g^{-1} \dot\wedge \delta g \dot\wedge g^{-1} \dot\wedge \mathrm{d}g + g^{-1} \dot\wedge \mathrm{d}\delta g \\ &= \omega^r \dot\wedge \breve{\omega}^r + \mathrm{d}\omega^r + g^{-1} \dot\wedge \mathrm{d}g \dot\wedge g^{-1} \dot\wedge \delta g \\ &= \mathrm{d}\omega^r + \omega^r \dot\wedge \breve{\omega}^r + \breve{\omega}^r \dot\wedge \omega^r \\ &= +\,\mathrm{d}\omega^r + [\omega^r \dot\wedge \breve{\omega}^r].\end{aligned}$$

Proof of the BRS relations. The first lines of (6.1l) now follow by addition of the first, resp. second lines of (6.14) and (6.15). Check of the second lines of (6.1l), resp. (6.1r): one has since $\delta^2 = 0$,

(6.19l)
$$
\begin{aligned}
s\omega^l &= s\delta g \dot{\wedge} g^{-1} - \delta g \dot{\wedge} sg^{-1} = -\delta g \dot{\wedge} sg^{-1} \\
&= -\delta g \dot{\wedge} g^{-1} \dot{\wedge} \delta g \dot{\wedge} g^{-1} = -\omega^l \dot{\wedge} \omega^l \\
&= -1/2\, [\omega^l \dot{\wedge} \omega^l]
\end{aligned}
$$

resp.

(6.19r)
$$
\begin{aligned}
s\omega^r &= sg^{-1} \dot{\wedge} \delta g + g^{-1} \dot{\wedge} s\delta g = sg^{-1} \dot{\wedge} \delta g \\
&= g^{-1} \dot{\wedge} \delta g \dot{\wedge} g^{-1} \dot{\wedge} \delta g = \omega^r \dot{\wedge} \omega^r = 1/2\, [\omega^r \wedge \omega^r].
\end{aligned}
$$

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